

# The Kitai Criterion and backward shifts

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## Abstract

It is proved that for any separable infinite dimensional Banach space  $X$ , there is a bounded linear operator  $T$  on  $X$  such that  $T$  satisfies the Kitai Criterion. The proof is based on quasimilarity argument and on showing that  $I + T$  satisfies the Kitai Criterion for certain backward weighted shifts  $T$ .

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## 1 Introduction

All vector spaces are assumed to be over  $\mathbb{K}$  being either the field  $\mathbb{C}$  of complex numbers or the field  $\mathbb{R}$  of real numbers. As usual,  $\mathbb{Z}$  is the set of integers,  $\mathbb{Z}_+$  is the set of non-negative integers and  $\mathbb{N}$  is the set of positive integers. For a Banach space  $X$ , symbol  $L(X)$  stands for the space of bounded linear operators on  $X$  and  $X^*$  is the space of continuous linear functionals on  $X$ .

**Definition 1.** Let  $X$  be a Banach space and  $T \in L(X)$ . We say that  $T$  is *hypercyclic* if there exists  $x \in X$  such that the orbit  $\{T^n x : n \in \mathbb{Z}_+\}$  is dense in  $X$ . Following Ansari [1],  $T$  is called *hereditarily hypercyclic* if for any infinite set  $\Lambda \subseteq \mathbb{Z}_+$ , there exists  $x \in X$  for which  $\{T^n x : n \in \Lambda\}$  is dense in  $X$ .

**Remark.** Hypercyclic operators have been intensely studied during last few decades, see surveys [14, 15] and references therein. It is also worth noting that in the terminology of [8, 13] hereditarily hypercyclic operators are called hereditarily hypercyclic with respect to the sequence  $n_k = k$  of all non-negative integers.

**Definition 2.** We say that a bounded linear operator  $T$  on a Banach space  $X$  satisfies the *Kitai Criterion* [16] if there exist two dense subsets  $E$  and  $F$  of  $X$  and a map  $S : F \rightarrow F$  such that  $TSy = y$ ,  $S^k y \rightarrow 0$  and  $T^k x \rightarrow 0$  as  $k \rightarrow \infty$  for any  $y \in F$  and  $x \in E$ .

**Definition 3.** Let  $X$  be a topological space. A continuous map  $T : X \rightarrow X$  is called *mixing* if for any two non-empty open sets  $U, V \subseteq X$ ,  $T^n(U) \cap V \neq \emptyset$  for all sufficiently large  $n \in \mathbb{N}$ .

It is well-known that a bounded linear operator  $T$  on a Banach space is mixing if and only if it is hereditarily hypercyclic, see [13]. Moreover, if  $T$  satisfies the Kitai Criterion, then  $T$  is mixing, see [13, 10]. Grivaux [13], answering a question raised by Shapiro, proved that for any separable infinite dimensional Banach space  $X$ , there exists a mixing  $T \in L(X)$ , that does not satisfy the Kitai criterion. We address the question raised by Petersson (Miniworkshop: Hypercyclicity and Linear Chaos, Oberwolfach, August, 2006), whether there is a bounded linear operator satisfying the Kitai Criterion on any separable infinite dimensional Banach space. The following theorem gives an affirmative answer to this question.

**Theorem 1.1.** *Let  $X$  be a separable infinite dimensional Banach space. Then there exists  $T \in L(X)$  satisfying the Kitai Criterion.*

This theorem fits into the following chain of results. Gerzog [12] proved that there is a supercyclic continuous linear operator on any separable infinite dimensional Banach space. Later Ansari [2] and Bernal-González [7] showed independently that for any separable infinite dimensional Banach space  $X$

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there is a hypercyclic operator  $T \in L(X)$ . Finally, as we have already mentioned, Grivaux [13] proved that there is a mixing operator on any separable infinite dimensional Banach space.

Recall that a *backward weighted shift* on  $\ell_p = \ell_p(\mathbb{Z}_+)$  for  $1 \leq p < \infty$  or  $c_0 = c_0(\mathbb{Z}_+)$  is the operator  $T$  acting on the canonical basis  $\{e_n\}_{n=0}^\infty$  as follows:  $Te_0 = 0$  and  $Te_n = w_n e_{n-1}$  for  $n \geq 1$ , where  $w = \{w_n\}_{n \in \mathbb{N}}$  is a bounded sequence of non-zero numbers in  $\mathbb{K}$ . The proof of Theorem 1.1 is based on the fact that operators  $I + T$  satisfy the Kitai Criterion for certain backward weighted shifts  $T$ . Namely, the following theorem holds.

**Theorem 1.2.** *Let  $X$  be either  $\ell_p$  with  $1 \leq p < \infty$  or  $X = c_0$  and  $T : X \rightarrow X$  be a backward weighted shift with the weight sequence  $\{w_n\}_{n \in \mathbb{N}}$ . Assume also that*

$$\lim_{n \rightarrow \infty} \left( n! \prod_{j=1}^n |w_j| \right)^{1/n} > 0. \quad (1.1)$$

*Then  $I + T$  satisfies the Kitai criterion.*

It is worth noting that if there exists  $c > 0$  such that  $|w_n| \geq c/n$  for each  $n \in \mathbb{N}$ , then (1.1) is satisfied. Thus, we have the following corollary.

**Corollary 1.3.** *Let  $X$  be either  $\ell_p$  with  $1 \leq p < \infty$  or  $X = c_0$  and  $T : X \rightarrow X$  be a backward weighted shift with the weight sequence  $\{w_n\}_{n \in \mathbb{N}}$ . Assume also that there exists  $c > 0$  such that  $|w_n| \geq c/n$  for each  $n \in \mathbb{N}$ . Then  $I + T$  satisfies the Kitai criterion.*

Recall that according to a theorem by Salas [19] the operator  $I + T$  is hypercyclic for any backward weighted shift  $T$  on  $\ell_p$  with  $1 \leq p < \infty$ . Grivaux [13] has observed that basically the same proof allows to show that the operators  $I + T$  in the Salas theorem are mixing. Theorem 1.2 shows that under certain conditions on the weight sequence these operators satisfy the Kitai criterion.

Theorems 1.1 and 1.2 are proved in Sections 2. Section 3 is devoted to concluding remarks.

## 2 Operators satisfying the Kitai Criterion

In this section we shall prove Theorems 1.1 and 1.2. We need some preparation.

### 2.1 Biorthogonal sequences

We are going to prove the existence of biorthogonal sequences with certain properties on any separable infinite dimensional Banach space. Recall that a family  $\{(x_\alpha, f_\alpha) : \alpha \in A\}$ , where  $x_\alpha$  are vectors in a Banach space  $X$  and  $f_\alpha \in X^*$ , is called *biorthogonal* if  $f_\alpha(x_\beta) = \delta_{\alpha,\beta}$  for any  $\alpha, \beta \in A$ ,  $\delta_{\alpha,\beta}$  being the Kronecker delta, that is  $\delta_{\alpha,\beta} = 1$  if  $\alpha = \beta$  and  $\delta_{\alpha,\beta} = 0$  if  $\alpha \neq \beta$ .

**Proposition 2.1.** *Let  $X$  be a separable infinite dimensional Banach space,  $\{b_k\}_{k \in \mathbb{Z}_+}$  be sequence of numbers in  $[3, \infty)$  such that  $b_k \rightarrow \infty$  as  $k \rightarrow \infty$  and  $\{n_k\}_{k \in \mathbb{Z}_+}$  be a strictly increasing sequence of positive integers such that  $n_0 = 0$  and  $n_{k+1} - n_k \geq 2$  for each  $k \in \mathbb{Z}_+$ . Then there exists a biorthogonal sequence  $\{(x_k, f_k)\}_{k \in \mathbb{Z}_+}$  in  $X \times X^*$  such that*

(B1)  $\|x_k\| = 1$  for each  $k \in \mathbb{Z}_+$ ;

(B2)  $\text{span}\{x_k : k \in \mathbb{Z}_+\}$  is dense in  $X$ ;

(B3)  $\|f_{n_k}\| \leq b_k$  for each  $k \in \mathbb{Z}_+$ ;

(B4)  $\|f_j\| \leq 3$  if  $j \in \mathbb{Z}_+ \setminus \{n_k : k \in \mathbb{Z}_+\}$ ;

(B5) for any  $k \in \mathbb{Z}_+$  and any numbers  $c_j \in \mathbb{K}$  with  $n_k + 1 \leq j \leq n_{k+1} - 1$ ,

$$\frac{1}{2} \left\| \sum_{j=n_k+1}^{n_{k+1}-1} c_j x_j \right\| \leq \left( \sum_{j=n_k+1}^{n_{k+1}-1} |c_j|^2 \right)^{1/2} \leq 2 \left\| \sum_{j=n_k+1}^{n_{k+1}-1} c_j x_j \right\|.$$

The proof of the above proposition is based on the following theorem, which is known as Dvoretzky theorem on almost spherical sections [11].

**Theorem D.** *For each  $n \in \mathbb{N}$  and each  $\varepsilon > 0$ , there exists  $m = m(n, \varepsilon) \in \mathbb{N}$  such that for any Banach space  $X$  with  $\dim X \geq m$  there is an  $n$ -dimensional linear subspace  $L$  in  $X$  and a linear basis  $e_1, \dots, e_n$  in  $L$  for which  $\|e_j\| = 1$  for  $1 \leq j \leq n$  and*

$$\frac{1}{1+\varepsilon} \left\| \sum_{j=1}^n c_j e_j \right\| \leq \left( \sum_{j=1}^n |c_j|^2 \right)^{1/2} \leq (1+\varepsilon) \left\| \sum_{j=1}^n c_j e_j \right\| \quad \text{for any } (c_1, \dots, c_n) \in \mathbb{K}^n.$$

We also need the following technical lemmas.

**Lemma 2.2.** *Let  $A$  be an infinite countable set and  $\{a_n\}_{n \in A}$  be a sequence of non-negative numbers such that there exists  $c > 0$  for which the set  $\{n \in A : a_n \leq c\}$  is infinite and  $\{b_n\}_{n \in \mathbb{Z}_+}$  be a sequence of numbers in  $[c, \infty)$  such that  $b_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then there exists a bijection  $\pi : \mathbb{Z}_+ \rightarrow A$  such that  $a_{\pi(n)} \leq b_n$  for each  $n \in \mathbb{Z}_+$ .*

*Proof.* Since the set  $\{n \in A : a_n \leq c\}$  is infinite, we can choose two disjoint infinite subsets  $B$  and  $C$  of  $A$  such that  $a_n \leq c$  for each  $n \in B$  and  $B \cup C = A$ . Fix a bijection  $\varphi : \mathbb{Z}_+ \rightarrow C$ . Since  $b_n \rightarrow \infty$  as  $n \rightarrow \infty$ , we can choose a sequence  $\{j_k\}_{k \in \mathbb{Z}_+}$  of non-negative integers such that  $j_{k+1} - j_k \geq 2$  and  $a_{\varphi(k)} \leq b_{j_k}$  for each  $k \in \mathbb{Z}_+$ . Denote  $D = \mathbb{Z}_+ \setminus \{j_k : k \in \mathbb{Z}_+\}$ . Since  $j_{k+1} - j_k \geq 2$ , the set  $D$  is infinite. Since both  $B$  and  $D$  are infinite and countable, there exists a bijection  $\psi : D \rightarrow B$ . Now consider the map  $\pi : \mathbb{Z}_+ \rightarrow A$  defined by the formula

$$\pi(n) = \begin{cases} \varphi(k) & \text{if } n = j_k, \\ \psi(n) & \text{if } n \in D. \end{cases}$$

Clearly  $\pi$  is a bijection. If  $n \in D$ , we have  $a_{\pi(n)} = a_{\psi(n)} \leq c \leq b_n$ . Indeed,  $a_m \leq c$  for  $m \in B$  and  $c \leq b_n$  for  $n \in \mathbb{Z}_+$ . If  $n = j_k$ , we have  $a_{\pi(n)} = a_{\varphi(k)} \leq b_{j_k} = b_n$ . Thus,  $a_{\pi(n)} \leq b_n$  for each  $n \in \mathbb{Z}_+$ .  $\square$

**Lemma 2.3.** *Let  $X$  be an infinite dimensional Banach space,  $u \in X$  and  $\{(x_k, f_k) : 1 \leq k \leq n\}$  be a finite biorthogonal sequence in  $X \times X^*$ . Then there exist  $x_{n+1} \in X$  and  $f_{n+1} \in X^*$  such that  $\|x_{n+1}\| = 1$ ,  $\{(x_k, f_k) : 1 \leq k \leq n+1\}$  is biorthogonal and  $u \in \text{span}\{x_1, \dots, x_{n+1}\}$ .*

*Proof.* Since  $X = L \oplus N$ , where  $L = \text{span}\{x_1, \dots, x_n\}$  and  $N = \bigcap_{k=1}^n \ker f_k$ , we can pick a vector  $x_{n+1} \in N$  such that  $\|x_{n+1}\| = 1$  and  $u \in M = \text{span}(L \cup \{x_{n+1}\}) = \text{span}\{x_1, \dots, x_{n+1}\}$ . Since  $x_1, \dots, x_{n+1}$  is a linear basis in  $M$ , there exists a unique linear functional  $g : M \rightarrow \mathbb{K}$  such that  $g(x_j) = 0$  for  $1 \leq j \leq n$  and  $g(x_{n+1}) = 1$ . According to the Hahn–Banach theorem, there exists  $f_{n+1} \in X^*$  such that  $f_{n+1}|_M = g$ . Clearly the pair  $(x_{n+1}, f_{n+1})$  satisfies all required conditions.  $\square$

**Lemma 2.4.** *Let  $m \in \mathbb{N}$ ,  $X$  be an infinite dimensional Banach space and  $\{(x_k, f_k) : 1 \leq k \leq n\}$  be a finite biorthogonal sequence in  $X \times X^*$ . Then there exist  $x_{n+1}, \dots, x_{n+m} \in X$  and  $f_{n+1}, \dots, f_{n+m} \in X^*$  such that  $\{(x_k, f_k) : 1 \leq k \leq n+m\}$  is biorthogonal,  $\|x_k\| = 1$  and  $\|f_k\| \leq 3$  for  $n+1 \leq k \leq n+m$  and*

$$\frac{1}{2} \left\| \sum_{j=n+1}^{n+m} c_j x_j \right\| \leq \left( \sum_{j=n+1}^{n+m} |c_j|^2 \right)^{1/2} \leq 2 \left\| \sum_{j=n+1}^{n+m} c_j x_j \right\| \quad \text{for any } (c_{n+1}, \dots, c_{n+m}) \in \mathbb{K}^m. \quad (2.1)$$

*Proof.* Fix  $\delta \in (0, 1/2)$  such that  $\frac{(1+\delta)(2-\delta)}{1-\delta} < 3$  and let  $L = \text{span}\{x_1, \dots, x_n\}$ ,  $S = \{x \in L : \|x\| = 1\}$ . Since  $L$  is finite-dimensional,  $S$  is a compact metric space with respect to the metric inherited from  $X$ . Hence we can find a finite set  $\{u_1, \dots, u_r\} \subset S$  such that for each  $x \in S$  there exists  $j \in \{1, \dots, r\}$  for which  $\|x - u_j\| < \delta$ . According to the Hahn–Banach theorem, there exist functionals  $\varphi_1, \dots, \varphi_r \in X^*$  such that  $\|\varphi_j\| = \varphi_j(u_j) = 1$  for  $1 \leq j \leq r$ . Consider the space

$$M = \bigcap_{l=1}^n \ker f_l \cap \bigcap_{j=1}^r \ker \varphi_j.$$

Since  $X$  is infinite dimensional,  $M$  is a closed infinite dimensional subspace of  $X$ . By Theorem D there exist  $x_{n+1}, \dots, x_{n+m} \in M$  such that  $\|x_j\| = 1$  for  $n+1 \leq j \leq n+m$  and

$$\frac{1}{1+\delta} \left\| \sum_{j=n+1}^{n+m} c_j x_j \right\| \leq \left( \sum_{j=n+1}^{n+m} |c_j|^2 \right)^{1/2} \leq (1+\delta) \left\| \sum_{j=n+1}^{n+m} c_j x_j \right\| \text{ for any } (c_{n+1}, \dots, c_{n+m}) \in \mathbb{K}^m. \quad (2.2)$$

Since  $\delta \leq 1/2$ , (2.2) implies (2.1).

Let now  $L_0 = \text{span}\{x_{n+1}, \dots, x_{n+m}\}$ . Since  $L_0 \subset M$  and the linear functionals  $f_j$  for  $1 \leq j \leq n$  vanish on  $M$ , we see that vectors  $x_1, \dots, x_{n+m}$  form a linear basis in  $L \oplus L_0$ . Thus, there exist unique linear functionals  $g_{n+1}, \dots, g_{n+m}$  on  $L \oplus L_0$  such that  $g_j(x_k) = \delta_{j,k}$  for  $n+1 \leq j \leq n+m$  and  $1 \leq k \leq n+m$ . We shall estimate the norms of  $g_j$ .

First, let  $y = \sum_{j=n+1}^{n+m} c_j x_j \in L_0$  and  $n+1 \leq k \leq n+m$ . Using (2.2), we obtain

$$|g_k(y)| = |c_k| \leq \left( \sum_{j=n+1}^{n+m} |c_j|^2 \right)^{1/2} \leq (1+\delta) \|y\|.$$

Thus,

$$|g_k(y)| \leq (1+\delta) \|y\| \text{ for each } y \in L_0. \quad (2.3)$$

Let now  $x \in L$  with  $\|x\| = 1$  and  $y \in L_0$ . Pick  $j \in \{1, \dots, r\}$  such that  $\|x - u_j\| \leq \delta$ . Then

$$\|x + y\| \geq |\varphi_j(x + y)| = |\varphi_j(x)| \geq |\varphi_j(u_j)| - |\varphi_j(x - u_j)| \geq 1 - \delta.$$

It follows that  $\|x + y\| \geq (1 - \delta) \|x\|$  for each  $x \in L$  and  $y \in L_0$ . Consequently

$$\|y\| \leq \|x\| + \|x + y\| \leq \left( 1 + \frac{1}{1 - \delta} \right) \|x + y\| = \frac{2 - \delta}{1 - \delta} \|x + y\| \text{ for each } x \in L \text{ and } y \in L_0. \quad (2.4)$$

Let now  $w \in L \oplus L_0$ . Then there exist unique  $x \in L$  and  $y \in L_0$  for which  $w = x + y$ . Applying (2.3) and (2.4), we obtain

$$|g_k(w)| = |g_k(x + y)| = |g_k(y)| \leq (1 + \delta) \|y\| \leq \frac{(1 + \delta)(2 - \delta)}{1 - \delta} \|w\|.$$

That is,  $\|g_k\| \leq \frac{(1 + \delta)(2 - \delta)}{1 - \delta} \leq 3$  for  $n + 1 \leq k \leq n + m$ . Using the Hahn–Banach theorem, we can now choose  $f_k \in X^*$  for  $n + 1 \leq k \leq n + m$  such that  $f_k|_{L \oplus L_0} = g_k$  and  $\|f_k\| \leq 3$ . It remains to observe that  $\{(x_k, f_k) : 1 \leq k \leq n + m\}$  is a biorthogonal sequence satisfying all desired conditions.  $\square$

*Proof of Proposition 2.1.* Since  $X$  is separable, we can choose a sequence  $\{u_k\}_{k \in \mathbb{Z}_+}$  of vectors in  $X$  such that  $\text{span}\{u_k : k \in \mathbb{Z}_+\}$  is dense in  $X$ . Denote  $m_k = n_k + k$  for  $k \in \mathbb{Z}_+$ . First, according to the Hahn–Banach theorem, we can pick  $(y_0, g_0) \in X \times X^*$  such that  $\|y_0\| = \|g_0\| = 1$  and  $u_0 \in \text{span}\{y_0\}$ , which will serve as the basis of an inductive process. Now we shall show inductively that for  $k = 0, 1, \dots$ , there exist  $\{(y_j, g_j) : m_k < j \leq m_{k+1}\} \subset X \times X^*$  such that

(P1)  $\{(y_j, g_j) : 0 \leq j \leq m_{k+1}\}$  is a biorthogonal sequence;

(P2)  $\|y_j\| = 1$  for  $m_k < j \leq m_{k+1}$ ;

(P3)  $u_{k+1} \in \text{span}\{y_0, \dots, y_{m_{k+1}}\}$ ;

(P4)  $\|g_j\| \leq 3$  for  $m_k < j < m_{k+1}$ ;

(P5) for any  $(c_{m_{k+1}}, \dots, c_{m_{k+1}-1}) \in \mathbb{K}^{m_{k+1}-m_k-1}$ ,

$$\frac{1}{2} \left\| \sum_{j=m_k+1}^{m_{k+1}-1} c_j y_j \right\| \leq \left( \sum_{j=m_k+1}^{m_{k+1}-1} |c_j|^2 \right)^{1/2} \leq 2 \left\| \sum_{j=m_k+1}^{m_{k+1}-1} c_j y_j \right\|.$$

Suppose  $(y_j, g_j)$  for  $j \leq m_k$  satisfying the desired properties are already constructed. Using Lemma 2.4, we find  $(y_j, g_j)$  for  $m_k < j < m_{k+1}$  in  $X \times X^*$  such that  $\{(y_j, g_j) : 0 \leq j < m_{k+1}\}$  is a biorthogonal sequence,  $\|y_j\| = 1$  for  $m_k < j < m_{k+1}$  and the conditions (P4) and (P5) are satisfied. By Lemma 2.3, we can now choose  $(y_{m_{k+1}}, g_{m_{k+1}}) \in X \times X^*$  such that  $\|y_{m_{k+1}}\| = 1$  and the conditions (P1) and (P3) are satisfied. Therefore  $\{(y_j, g_j) : m_k < j \leq m_{k+1}\}$  satisfy (P1–P5).

Thus we have just defined an inductive procedure, which provides a sequence  $\{(y_j, g_j) : j \in \mathbb{Z}_+\}$  of elements of  $X \times X^*$  satisfying

(Q1)  $\{(y_j, g_j) : j \in \mathbb{Z}_+\}$  is a biorthogonal sequence;

(Q2)  $\|y_j\| = 1$  for  $j \in \mathbb{Z}_+$ ;

(Q3)  $\text{span}\{u_j : j \in \mathbb{Z}_+\} \subseteq \text{span}\{y_j : j \in \mathbb{Z}_+\}$ ;

(Q4)  $\|g_j\| \leq 3$  for  $m_k < j < m_{k+1}$ ,  $k \in \mathbb{Z}_+$ ;

(Q5) for any  $k \in \mathbb{Z}_+$  and  $(c_{m_k+1}, \dots, c_{m_{k+1}-1}) \in \mathbb{K}^{m_{k+1}-m_k-1}$ ,

$$\frac{1}{2} \left\| \sum_{j=m_k+1}^{m_{k+1}-1} c_j y_j \right\| \leq \left( \sum_{j=m_k+1}^{m_{k+1}-1} |c_j|^2 \right)^{1/2} \leq 2 \left\| \sum_{j=m_k+1}^{m_{k+1}-1} c_j y_j \right\|.$$

Now we shall show that a biorthogonal sequence with the desired properties can be obtained as a permutation of the biorthogonal sequence  $\{(y_j, g_j)\}$ . Let

$$A = B \cup C, \quad \text{where} \quad B = \{m_k : k \in \mathbb{Z}_+\} \quad \text{and} \quad C = \{m_k + 1 : k \in \mathbb{Z}_+\}.$$

Clearly  $A$  is an infinite countable set. According to (Q4)  $\|g_j\| \leq 3$  if  $j \in B$ . By Lemma 2.2, there exists a bijection  $\varphi : \mathbb{Z}_+ \rightarrow A$  such that  $\|g_{\varphi(j)}\| \leq b_j$  for each  $j \in \mathbb{Z}_+$ . Now we consider the map  $\pi : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  defined by the formula

$$\pi(j) = \begin{cases} \varphi(k) & \text{if } j = n_k, \\ m_k + l + 1 & \text{if } j = n_k + l, 1 \leq l < n_{k+1} - n_k. \end{cases}$$

Using the equalities  $m_k = n_k + k$  and  $m_{k+1} - m_k = n_{k+1} - n_k + 1$ , it is easy to verify that  $\pi$  maps bijectively  $\mathbb{Z}_+ \setminus \{n_k : k \in \mathbb{Z}_+\}$  onto  $\mathbb{Z}_+ \setminus A$ . Since  $\varphi : \mathbb{Z}_+ \rightarrow A$  is also bijective, we have that  $\pi : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  is a bijection. Thus,  $\{(x_j, f_j) : j \in \mathbb{Z}_+\}$  with  $(x_j, f_j) = (y_{\pi(j)}, g_{\pi(j)})$  is just a permutation of the biorthogonal sequence  $\{(y_j, g_j) : j \in \mathbb{Z}_+\}$ . Hence  $\{(x_j, f_j) : j \in \mathbb{Z}_+\}$  is also a biorthogonal sequence and the properties (B1) and (B2) follow immediately from (Q2) and (Q3). Since  $\pi(\{n_k + 1, \dots, n_{k+1} - 1\}) = \{m_k + 2, \dots, m_{k+1} - 1\}$  for each  $k \in \mathbb{Z}_+$ , properties (B4) and (B5) follow from (Q4) and (Q5) respectively. Finally  $\|f_{n_k}\| = \|g_{\varphi(k)}\| \leq b_k$  for each  $k \in \mathbb{Z}_+$  and (B3) is also satisfied.  $\square$

We need Proposition 2.1 for one purpose only. Namely, we need it in order to prove the following lemma which is one of the two main ingredients in the proof of Theorem 1.1.

**Lemma 2.5.** *Let  $a \in \ell_2(\mathbb{N})$  and  $T_a$  be the weighted backward shift on  $\ell_1$  with the weight sequence  $a$ , that is,  $T_a e_0 = 0$  and  $T_a e_n = a_n e_{n-1}$  for  $n \geq 1$ , where  $\{e_n\}_{n \in \mathbb{Z}_+}$  is the canonical basis in  $\ell_1$ . Let also  $X$  be a separable infinite dimensional Banach space. Then there exist  $T \in L(X)$  and an injective bounded linear operator  $J : \ell_1 \rightarrow X$  with dense range such that  $JT_a = TJ$ .*

*Proof.* Since  $a \in \ell_2(\mathbb{N})$ , we can choose a sequence  $\{n_k : k \in \mathbb{Z}_+\}$  of non-negative integers and a sequence  $\{b_k : k \in \mathbb{Z}_+\}$  of numbers in  $[3, \infty)$  such that  $n_0 = 0$ ,  $n_{k+1} - n_k \geq 2$  for each  $k \in \mathbb{Z}_+$ ,  $b_k \rightarrow \infty$  as  $k \rightarrow \infty$ ,

$$\sum_{k=0}^{\infty} b_k |a_{n_k}| < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \left( \sum_{j=n_k+1}^{n_{k+1}} |a_j|^2 \right)^{1/2} < \infty. \quad (2.5)$$

Let  $\{(x_j, f_j) : j \in \mathbb{Z}_+\}$  be the biorthogonal system in  $X \times X^*$  furnished by Proposition 2.1 with the above choice of sequences  $\{n_k\}$  and  $\{b_k\}$ . Since  $\|x_n\| = 1$  for each  $n \in \mathbb{Z}_+$ , the formula

$$Ju = \sum_{n=0}^{\infty} u_n x_n$$

defines a bounded linear operator from  $\ell_1$  to  $X$ . Since  $f_j(Ju) = u_j$  for any  $j \in \mathbb{Z}_+$  and  $u \in \ell_1$ , we see that  $J$  is injective. Since  $J(e_j) = x_j$  for each  $j \in \mathbb{Z}_+$  and  $\text{span}\{x_j : j \in \mathbb{Z}_+\}$  is dense in  $X$ , the operator  $J$  has dense range. For each  $k \in \mathbb{Z}_+$ , we consider  $S_k \in L(X)$  defined by the formula

$$S_k x = \sum_{j=n_k+2}^{n_{k+1}-1} a_j f_j(x) x_{j-1},$$

assuming that  $S_k = 0$  if  $n_k - n_{k-1} = 2$ . Applying Proposition 2.1 (B5) and (B4), we have

$$\|S_k x\|^2 \leq 4 \sum_{j=n_k+2}^{n_{k+1}-1} |a_j|^2 |f_j(x)|^2 \leq 36 \|x\|^2 \sum_{j=n_k+2}^{n_{k+1}-1} |a_j|^2 \leq 36 \|x\|^2 \sum_{j=n_k+1}^{n_{k+1}} |a_j|^2.$$

Hence

$$\|S_k\| \leq 6 \left( \sum_{j=n_k+1}^{n_{k+1}} |a_j|^2 \right)^{1/2}.$$

Thus applying the second inequality in (2.5), we see that  $\sum_{k=0}^{\infty} \|S_k\| < \infty$  and therefore the formula

$$T_0 x = \sum_{k=0}^{\infty} S_k x$$

defines a bounded linear operator on  $X$ . Since by Proposition 2.1 (B3),  $\|f_{n_k}\| \leq b_k$ , we, using the first inequality in (2.5), see that the formula

$$T_1 x = \sum_{k=1}^{\infty} a_{n_k} f_{n_k}(x) x_{n_k-1}$$

also defines a bounded linear operator on  $X$ . Finally, the second inequality in (2.5) implies that  $\sum_{k=0}^{\infty} |a_{n_k+1}| < \infty$ . Thus, since  $\|f_{n_k+1}\| \leq 3$  according to Proposition 2.1 (B4), the formula

$$T_2 x = \sum_{k=0}^{\infty} a_{n_k+1} f_{n_k+1}(x) x_{n_k}$$

defines a bounded linear operator on  $X$  as well. Consider now the operator  $T = T_0 + T_1 + T_2 \in L(X)$ . It is straightforward to verify that  $Tx_0 = 0$  and  $Tx_n = a_n x_{n-1}$  for  $n \geq 1$ . Taking into account that  $Je_n = x_n$  for each  $n \in \mathbb{Z}_+$ , we see that  $JT_a e_0 = TJe_0 = 0$  and  $JT_a e_n = TJe_n = a_n x_{n-1}$  for  $n \geq 1$ . Hence  $JT_a = TJ$ .  $\square$

## 2.2 Orbits of $I - cV$ and $(I + cV)^{-1}$

Let  $V$  be the classical Volterra operator acting on  $L_2[0, 1]$ :

$$Vf(t) = \int_0^t f(s) ds.$$

We need the following fact on the behavior of the orbits of  $I - cV$  and  $(I + cV)^{-1}$  for  $c > 0$ . For the case  $c = 1$ , the proof can be found in [18] and the proof for general  $c > 0$  goes along the same lines. We include the proof for sake of completeness.

**Lemma 2.6.** *Let  $c > 0$ . Then for any  $f \in L_2[0, 1]$ ,*

$$\lim_{n \rightarrow \infty} \|(I - cV)^n f\| = 0 \tag{2.6}$$

$$\text{and } \lim_{n \rightarrow \infty} \|(I + cV)^{-n} f\| = 0. \tag{2.7}$$

*Proof.* First, take the function  $\mathbf{1}$  being identically 1. One can easily verify that  $(I - cV)^n \mathbf{1}(t) = L_n(ct)$ , where  $L_n$  are the Laguerre polynomials:

$$L_n(t) = \sum_{k=0}^n \frac{n!(-t)^k}{(n-k)!(k!)^2}.$$

From the well-known asymptotic formulae [20, Chapter 8] for Laguerre polynomials it follows that for any  $0 < a < b < \infty$ ,  $\|L_n\|_{L_\infty[0,b]} = O(1)$  and  $\|L_n\|_{L_\infty[a,b]} = O(n^{-1/2})$  as  $n \rightarrow \infty$ . Since  $(I - cV)^n \mathbf{1}(t) = L_n(ct)$ , we immediately obtain  $(I - cV)^n \mathbf{1} \rightarrow 0$  in  $L_2[0, 1]$  as  $n \rightarrow \infty$ . Now, let  $p(t) = \sum_{j=0}^n p_j t^j$  be any polynomial. Then  $p = q(V)\mathbf{1}$ , where  $q(t) = \sum_{j=0}^n j! p_j t^j$ . Therefore

$$(I - cV)^n p = (I - cV)^n q(V)\mathbf{1} = q(V)(I - cV)^n \mathbf{1} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

since  $(I - cV)^n \mathbf{1} \rightarrow 0$  and  $q(V)$  is a bounded linear operator on  $L_2[0, 1]$ . Since the space of polynomials is dense in  $L_2[0, 1]$ , we see that (2.6) is satisfied for  $f$  from a dense set.

Next,  $V + V^* = P$ , where  $P$  is the orthoprojection onto the one-dimensional space of constant functions. Hence

$$\operatorname{Re} \langle Vf, f \rangle = \frac{1}{2} \langle (V + V^*)f, f \rangle = \frac{1}{2} \langle Pf, f \rangle \geq 0 \quad \text{for any } f \in L_2[0, 1].$$

Thus, for any  $f \in L_2[0, 1]$  with  $\|f\| = 1$ , we have

$$\|(I + cV)f\| \geq \operatorname{Re} \langle f + cVf, f \rangle \geq \langle f, f \rangle = 1.$$

Hence  $(I + cV)^{-1}$  is a contraction. One can also easily verify that  $I - cV = M_c(I + cV)^{-1}M_c^{-1}$ , where  $M_c f(t) = e^{-ct}f(t)$ . It follows that  $I - cV$  and  $(I + cV)^{-1}$  are similar. Therefore, the operator  $I - cV$  is power bounded since it is similar to a contraction. Since  $I - cV$  is power bounded and (2.6) is satisfied for  $f$  from a dense set, we obtain that (2.6) is satisfied for all  $f \in L_2[0, 1]$ . Finally similarity of  $I - cV$  and  $(I + cV)^{-1}$  implies that (2.7) is also satisfied for any  $f \in L_2[0, 1]$ .  $\square$

### 2.3 The Kitai Criterion and quasisimilarity

It has been noticed by many authors that for any operator satisfying a certain cyclicity property (like being cyclic, supercyclic, hypercyclic, satisfying the Hypercyclicity Criterion or satisfying the Kitai Criterion), any other operator being in certain quasisimilarity relation with the first one satisfies the same property. The following lemma is proved in [17].

**Lemma 2.7.** *Let  $X$  and  $X_0$  be Banach spaces,  $T_0 \in L(X_0)$  and  $T \in L(X)$ . Suppose also that  $T_0$  satisfies the Kitai Criterion and there exists an injective bounded linear operator  $J : X_0 \rightarrow X$  with dense range such that  $JT_0 = TJ$ . Then  $T$  satisfies the Kitai Criterion.*

### 2.4 The Kitai Criterion for $I + T$ , where $T$ is a backward weighted shift

**Remark.** Let  $w = \{w_n\}_{n \in \mathbb{N}}$  and  $u = \{u_n\}_{n \in \mathbb{N}}$  be two bounded sequence of non-zero numbers in  $\mathbb{K}$  such that  $|w_n| = |u_n|$  for each  $n \in \mathbb{N}$  and  $T_w, T_u$  be backward weighted shifts with weight sequences  $w$  and  $u$  respectively acting on  $X$  being either  $\ell_p$  for  $1 \leq p < \infty$  or  $c_0$ . Then the operators  $T_w$  and  $T_u$  are isometrically similar. Indeed, consider the sequence  $\{d_n\}_{n \in \mathbb{Z}_+}$  defined as  $d_0 = 1$  and  $d_n = \prod_{j=1}^n \frac{w_j}{u_j}$  for  $n \geq 1$ . Then  $|d_n| = 1$  for each  $n \in \mathbb{Z}_+$  and therefore the diagonal operator  $D \in L(X)$ , which acts on the basic vectors according to the formula  $De_n = d_n e_n$  for  $n \in \mathbb{Z}_+$ , is an isometric isomorphism of  $X$  onto itself. One can easily verify that  $T_u = D^{-1}T_w D$ . That is,  $T_w$  and  $T_u$  are isometrically similar.

For a bounded linear operator  $S$  on a Banach space  $X$  we denote

$$\mathcal{E}(S) = \{x \in X : S^n x \rightarrow 0 \text{ as } n \rightarrow \infty\}. \quad (2.8)$$

It follows immediately from Definition 2 that an invertible  $S \in L(X)$  satisfies the Kitai Criterion if and only if both  $\mathcal{E}(S)$  and  $\mathcal{E}(S^{-1})$  are dense in  $X$ .

**Lemma 2.8.** *Let  $\lambda \in \mathbb{K} \setminus \{0\}$  and  $T$  be the weighted backward shift on  $\ell_1$  with the weight sequence  $w_n = 1/n$ ,  $n \in \mathbb{N}$ . Then the operator  $I + \lambda T$  satisfies the Kitai criterion.*

*Proof.* Let  $c = |\lambda|/2$ . Since  $T$  is quasinilpotent, the operator  $I + \lambda T$  is invertible and therefore it suffices to prove that  $\mathcal{E}(I + \lambda T)$  and  $\mathcal{E}((I + \lambda T)^{-1})$  are dense in  $\ell_1$ . According to the above remark the operators  $\lambda T$ ,  $2cT$  and  $-2cT$  are isometrically similar. Hence  $I + \lambda T$  is similar to  $I - 2cT$  and  $(I + \lambda T)^{-1}$  is similar to  $(I + 2cT)^{-1}$ . Thus, it suffices to prove that

$$\mathcal{E}(I - 2cT) \text{ is dense in } \ell_1 \quad (2.9)$$

$$\text{and } \mathcal{E}((I + 2cT)^{-1}) \text{ is dense in } \ell_1. \quad (2.10)$$

Consider the bounded linear operator  $J_0 : \ell_\infty \rightarrow L_2[0, 1]$  defined by the formula

$$J_0 x(t) = \sum_{n=0}^{\infty} \frac{x_n (1-t)^n}{2^n}.$$

Naturally identifying  $\ell_\infty$  with  $\ell_1^*$  and  $(L_2[0, 1])^*$  with  $L_2[0, 1]$ , one can easily see that  $J_0$  is  $*$ -weakly continuous and therefore  $J_0 = J^{**}$ , where  $J = J_0|_{c_0}$ . Uniqueness theorem for analytic functions implies that  $J_0 = J^{**}$  is injective and therefore  $J^* : L_2[0, 1] \rightarrow \ell_1$  has dense range. Clearly  $J(c_0)$  contains all polynomials and therefore is dense in  $L_2[0, 1]$ . Hence  $J^*$  is injective. A direct calculation shows that the dual of the Volterra operator  $V$  on  $L_2[0, 1]$  acts according to the formula

$$V^* f(t) = \int_t^1 f(s) ds.$$

Consider also the weighted forward shift  $S \in L(c_0)$ ,  $Se_n = e_{n+1}/(n+1)$  for  $n \in \mathbb{Z}_+$ , where  $\{e_n\}_{n \in \mathbb{Z}_+}$  is the canonical basis of  $c_0$ . We shall verify that

$$2JS = V^*J. \quad (2.11)$$

Indeed, for any  $n \in \mathbb{Z}_+$ ,

$$(2JS e_n)(t) = \frac{2(Je_{n+1})(t)}{n+1} = \frac{2(1-t)^{n+1}}{(n+1)2^{n+1}} = \frac{(1-t)^{n+1}}{(n+1)2^n}$$

$$\text{and } (V^*J e_n)(t) = \int_t^1 \frac{(1-s)^n}{2^n} ds = \frac{(1-t)^{n+1}}{(n+1)2^n}.$$

Thus,  $2JS e_n = V^*J e_n$  for each  $n \in \mathbb{Z}_+$  and (2.11) follows. Taking the adjoint of the both sides of the equality (2.11) and taking into account that  $S^* = T$ , we obtain  $2TJ^* = J^*V$ , which immediately implies

$$(I - 2cT)J^* = J^*(I - cV) \quad \text{and} \quad (I + 2cT)^{-1}J^* = J^*(I + cV)^{-1}. \quad (2.12)$$

Applying Lemma 2.6, we see that for any  $f \in L_2[0, 1]$ ,

$$(I - 2cT)^n J^* f = J^*(I - cV)^n f \rightarrow 0 \quad \text{and} \quad (I + 2cT)^{-n} J^* f = J^*(I + cV)^{-n} f \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence each of the spaces  $\mathcal{E}(I - 2cT)$  and  $\mathcal{E}((I + 2cT)^{-1})$  contains  $J^*(L_2[0, 1])$ , which is dense in  $\ell_1$ . Thus, (2.9) and (2.10) hold and therefore  $I + \lambda T$  satisfies the Kitai Criterion.  $\square$

## 2.5 Proof of Theorem 1.2

Condition (1.1) implies that there exists  $c > 0$  for which the sequence

$$d_0 = 1, \quad d_n = \prod_{j=1}^n \frac{c}{n w_n} \quad \text{for } n \in \mathbb{N}$$

is bounded. Let  $S$  be the weighted backward shift on  $\ell_1$  with the weight sequence  $w_n = 1/n$ ,  $n \in \mathbb{N}$ ,  $\{e_n\}_{n \in \mathbb{Z}_+}$  be the canonical basis in  $X$  and  $\{e'_n\}_{n \in \mathbb{Z}_+}$  be the canonical basis in  $\ell_1$ . Since  $\{d_n\}_{n \in \mathbb{Z}_+}$  is bounded there exists a unique bounded linear operator  $J : \ell_1 \rightarrow X$  such that  $J e'_n = d_n e_n$  for  $n \in \mathbb{Z}_+$ . Since  $d_n \neq 0$  for each  $n \in \mathbb{Z}_+$ , the operator  $J$  is injective and has dense range. Using the definition of  $d_n$ , one can easily verify that  $TJ = cJS$  and therefore  $(I + T)J = J(I + cS)$ . By Lemma 2.8  $I + cS$  satisfies the Kitai Criterion. From Lemma 2.7 it follows now that  $I + T$  also satisfies the Kitai Criterion.



## 2.6 Proof of Theorem 1.1

Let  $S$  be the weighted backward shift on  $\ell_1$  with the weight sequence  $w_n = 1/n$ ,  $n \in \mathbb{N}$ . By Lemma 2.5, there exist  $T \in L(X)$  and an injective bounded linear operator  $J : \ell_1 \rightarrow X$  with dense range such that  $JS = TJ$ . Hence  $J(I+S) = (I+T)J$ . By Lemma 2.8  $I+S$  satisfies the Kitai Criterion. Now Lemma 2.7 implies that  $I+T$  also satisfies the Kitai Criterion. The proof is complete.

## 3 Concluding remarks

The operator, we construct in the proof of Theorem 1.1, is the sum of the identity operator and a compact quasinilpotent operator. In particular, its spectrum is the one-point set  $\{1\}$ . It is worth noting that there are Banach spaces, where one can not expect anything else. Namely, if  $X$  is a hereditarily indecomposable Banach space [3], then the spectrum of any hypercyclic operator  $T$  on  $X$  is a one-point set  $\{\lambda\}$  with  $|\lambda| = 1$ , see, for instance, [13].

The sufficient condition of frequent hypercyclicity [6], an interesting concept recently introduced by Bayart and Grivaux, is related to a stronger form of the Kitai Criterion. Let us say that a bounded linear operator  $T$  on a Banach space  $X$  satisfies the *strong Kitai Criterion* if the spaces  $E$  and  $F$  in Definition 2 may be chosen to be the same space:  $E = F$ . Clearly, an invertible operator satisfies the strong Kitai Criterion if and only if  $\mathcal{E}(T) \cap \mathcal{E}(T^{-1})$  is dense in  $X$ . We shall immediately see that this condition is much more restrictive than the Kitai Criterion.

**Lemma 3.1.** *Let  $T$  be a bounded linear operator on a complex Banach space  $X$  and  $\sigma(T) = \{1\}$ . Then  $\mathcal{E}(T) \cap \mathcal{E}(T^{-1}) = \{0\}$ .*

*Proof.* Let  $Y$  be the space  $\mathcal{E}(T) \cap \mathcal{E}(T^{-1})$  endowed with the norm  $\|y\|_Y = \max\{\|T^n y\| : n \in \mathbb{Z}\}$ . It is straightforward to see that  $Y$  is a Banach space,  $T(Y) \subseteq Y$  and the restriction  $A = T|_Y$ , considered as an operator acting on the Banach space  $Y$  is an invertible isometry. Now, if  $z \in \mathbb{C} \setminus \{1\}$ , then, using the fact that  $T - I$  is quasinilpotent, we see that  $(T - zI)^{-1} = (1 - z)^{-1} \sum_{k=0}^{\infty} (1 - z)^k (T - I)^k$ , where the series in the right-hand side is operator norm absolutely convergent. Note that if  $S$  is a bounded linear operator on  $X$  such that  $ST = TS$ , then  $S(Y) \subseteq Y$  and the restriction of  $S$  to  $Y$ , considered as an operator on  $Y$ , has the operator norm not exceeding  $\|S\|$ . Hence, the series  $(1 - z)^{-1} \sum_{k=0}^{\infty} (1 - z)^k (A - I)^k$  is operator norm absolutely convergent to  $(A - zI)^{-1}$ . Thus,  $z \notin \sigma(A)$ . Hence,  $\sigma(A) = \{1\}$ . Since  $A$  is an isometry, we have that  $\{\|A^n\|\}_{n \in \mathbb{Z}}$  is bounded. A classical theorem due to Gel'fand asserts that an invertible element  $x$  of a unital Banach algebra is the identity if  $\sigma(x) = \{1\}$  and  $\{\|x^n\|\}_{n \in \mathbb{Z}}$  is bounded. Thus,  $A = I$  and therefore  $Ty = y$  for any  $y \in Y$ . From the definition of  $Y$  it follows that  $Ty = y$  for  $y \in Y$  happens if and only if  $y = 0$ . Thus,  $Y = \{0\}$ .  $\square$

**Proposition 3.2.** *Let  $X$  be a hereditarily indecomposable Banach space. Then there is no bounded linear operators on  $X$  satisfying the strong Kitai Criterion.*

*Proof.* The case  $\mathbb{K} = \mathbb{R}$  reduces to the case  $\mathbb{K} = \mathbb{C}$  by passing to the complexification. Thus, we can assume that  $\mathbb{K} = \mathbb{C}$ . Let  $S$  be a bounded linear operator on  $X$ , satisfying the strong Kitai Criterion. As we have already mentioned, the fact that  $X$  is hereditarily indecomposable implies that  $\sigma(S)$  is a one-point set  $\{\lambda\}$  with  $|\lambda| = 1$ . Then  $T = \lambda^{-1}S$  satisfies the strong Kitai Criterion and  $\sigma(T) = \{1\}$ . Since  $T$  is invertible and satisfies the strong Kitai Criterion, we have that  $\mathcal{E}(T) \cap \mathcal{E}(T^{-1})$  is dense in  $X$ , which is not possible according to Lemma 3.1.  $\square$

It is also worth noting that Theorem 1.2 is surprisingly sharp. The following observation is due to Atzmon, see [4, 5].

**Theorem A.** *Let  $k \in \mathbb{N}$  and  $T$  be a bounded linear operator on a Banach space  $X$  such that  $\|T^n\|^{1/n} = o(1/n)$  as  $n \rightarrow \infty$ . Then for  $x \in X$ ,  $\|(I+T)^n x\| = O(n^k)$  as  $n \rightarrow \infty$  if and only if  $T^k x = 0$ .*

From Theorem A it immediately follows that  $\mathcal{E}(I+T) = \{0\}$  if  $\|T^n\|^{1/n} = o(1/n)$  as  $n \rightarrow \infty$ . Indeed, if  $x \in \mathcal{E}(I+T)$  then by Theorem A we have  $Tx = 0$  and therefore  $(I+T)^n x = x$  for each  $n \in \mathbb{Z}_+$ . Taking into account that  $\mathcal{E}(S)$  is dense for each operator  $S$  satisfying the Kitai Criterion, we have the following corollary.

**Corollary 3.3.** *Let  $T$  be a bounded linear operator on a Banach space  $X$  such that  $\|T^n\|^{1/n} = o(1/n)$  as  $n \rightarrow \infty$ . Then  $I + T$  does not satisfy the Kitai Criterion.*

From Corollary 3.3 it follows, in particular, that if  $T$  is a weighted backward shift on  $\ell_p$  for  $1 \leq p < \infty$  or on  $c_0$  with the weight sequence  $\{w_n\}$  satisfying  $w_n = o(n^{-1})$  as  $n \rightarrow \infty$ , then  $I + T$  does not satisfy the Kitai Criterion.

It is well-known and easy to see that the spectrum of a backward weighted shift  $T$  is always the disk  $\{z \in \mathbb{C} : |z| \leq r\}$ , where  $r \geq 0$  is the spectral radius of  $T$ . As it easily follows from the results of Chan and Shapiro [9],  $I + T$  satisfies the Kitai Criterion if  $r > 0$ . From this point of view Theorem 1.2 gives a much more subtle sufficient condition for such operators to satisfy the Kitai Criterion. In particular, it shows that there are many quasinilpotent backward weighted shifts  $T$  for which  $I + T$  satisfies the Kitai Criterion.

Finally observe that the Kitai Criterion makes sense for operators on arbitrary topological vector spaces. A natural question arises in relation with Theorem 1.1. Namely, does there exist a continuous linear operator satisfying the Kitai Criterion on any separable infinite dimensional Fréchet space?

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